12.1 Introduction to Limits

The concept of a limit is useful in applications involving maximization. For example, in Exercise 3 on page 826, you will use the concept of a limit to verify the maximum volume of an open box.

- **Understand the limit concept.**
- **Use the definition of a limit to estimate limits.**
- **Determine whether limits of functions exist.**
- **Use properties of limits and direct substitution to evaluate limits.**

The Limit Concept

The notion of a limit is a *fundamental* concept of calculus. In this chapter, you will learn how to evaluate limits and how to use them in the two basic problems of calculus: the tangent line problem and the area problem.

EXAMPLE 1 **Finding a Rectangle of Maximum Area**

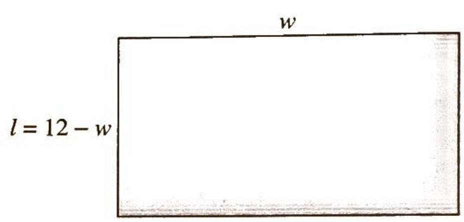
Find the dimensions of a rectangle with perimeter 24 inches that yield the maximum area.

Solution Let w represent the width of the rectangle and let l represent the length of the rectangle. From

$$2w + 2l = 24 \quad \text{Perimeter is 24.}$$

it follows that $l = 12 - w$, as shown in the figure below. So, the area of the rectangle is

$$A = lw \quad \text{Formula for area}$$
$$= (12 - w)w \quad \text{Substitute } 12 - w \text{ for } l.$$
$$= 12w - w^2. \quad \text{Simplify.}$$



Using this model for area, experiment with different values of w to see how to obtain the maximum area. After checking several values, it appears that the maximum area occurs when $w = 6$, as shown in the table.

Width, w	5.0	5.5	5.9	6.0	6.1	6.5	7.0
Area, A	35.00	35.75	35.99	36.00	35.99	35.75	35.00

In limit terminology, you say "the limit of A as w approaches 6 is 36" and write

$$\lim_{w \to 6} A = \lim_{w \to 6} (12w - w^2) = 36.$$

So, the dimensions of a rectangle with perimeter 24 inches that yield the maximum area are $w = 6$ inches and $l = 12 - 6 = 6$ inches.

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Find the dimensions of a rectangle with perimeter 52 inches that yield the maximum area.

REMARK An alternative notation for $\lim_{x \to c} f(x) = L$ is

$$f(x) \to L \text{ as } x \to c$$

which is read as "$f(x)$ approaches L as x approaches c."

Definition of a Limit

Definition of a Limit

If $f(x)$ becomes arbitrarily close to a unique number L as x approaches c from either side, then the **limit** of $f(x)$ as x approaches c is L. This is written as

$$\lim_{x \to c} f(x) = L.$$

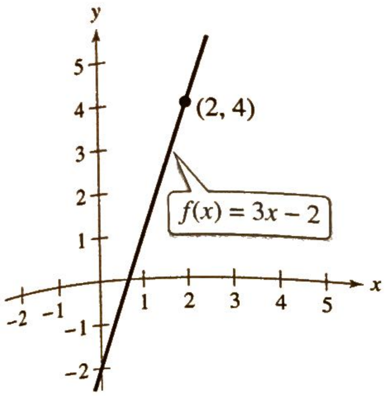

Figure 12.1

EXAMPLE 2 Estimating a Limit Numerically

Use a table to estimate the limit numerically: $\lim_{x \to 2} (3x - 2)$.

Solution Let $f(x) = 3x - 2$. Then construct a table that shows values of $f(x)$ for two sets of x-values—one that approaches 2 from the left and one that approaches 2 from the right.

x	1.9	1.99	1.999	2.0	2.001	2.01	2.1
$f(x)$	3.700	3.970	3.997	?	4.003	4.030	4.300

From the table, it appears that the closer x gets to 2, the closer $f(x)$ gets to 4. So, estimate the limit to be 4. Figure 12.1 verifies this conclusion.

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Use a table to estimate the limit numerically: $\lim_{x \to 3} (3 - 2x)$.

In Figure 12.1, note that the graph of $f(x) = 3x - 2$ is continuous. For graphs that are not continuous, finding a limit can be more challenging.

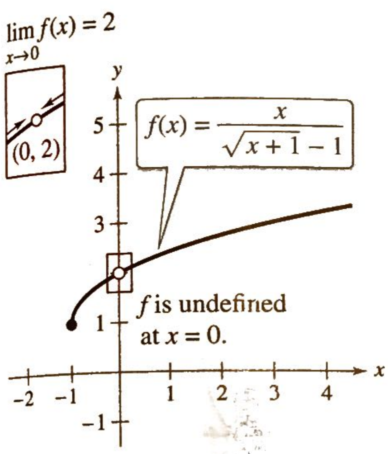

Figure 12.2

EXAMPLE 3 Estimating a Limit Numerically

Use a table to estimate the limit numerically.

$$\lim_{x \to 0} \frac{x}{\sqrt{x + 1} - 1}$$

Solution Let $f(x) = x/\left(\sqrt{x + 1} - 1\right)$. Then construct a table that shows values of $f(x)$ for two sets of x-values—one that approaches 0 from the left and one that approaches 0 from the right.

x	−0.01	−0.001	−0.0001	0	0.0001	0.001	0.01
$f(x)$	1.99499	1.99950	1.99995	?	2.00005	2.00050	2.00499

From the table, it appears that the limit is 2. Figure 12.2 verifies this conclusion.

REMARK In Example 3, note that $f(0)$ is undefined, so it is not possible to *reach* the limit. In Example 2, note that $f(2) = 4$, so it *is* possible to reach the limit.

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Use a table to estimate the limit numerically.

$$\lim_{x \to 1} \frac{x - 1}{x^2 + 3x - 4}$$

In Example 3, note that $f(x)$ has a limit when $x \to 0$ even though the function is not defined when $x = 0$. This often happens, and it is important to realize that *the existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as x approaches c.*

EXAMPLE 4 Estimating a Limit

Estimate the limit: $\lim_{x \to 1} \frac{x^3 - x^2 + x - 1}{x - 1}$.

Numerical Solution

Let $f(x) = (x^3 - x^2 + x - 1)/(x - 1)$. Then construct a table that shows values of $f(x)$ for two sets of x-values—one that approaches 1 from the left and one that approaches 1 from the right.

x	0.9	0.99	0.999	1.0
$f(x)$	1.8100	1.9801	1.9980	?

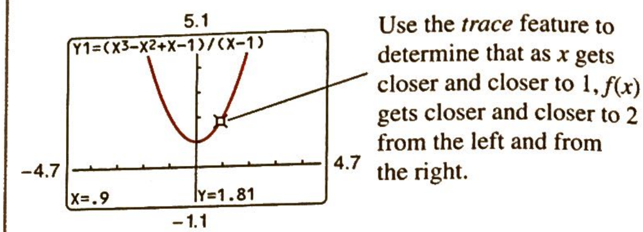
x	1.001	1.01	1.1
$f(x)$	2.0020	2.0201	2.2100

From the table, it appears that the limit is 2.

Graphical Solution

Use a graphing utility to graph

$$f(x) = (x^3 - x^2 + x - 1)/(x - 1).$$



From the graph, estimate the limit to be 2. As you use the *trace* feature, notice that there is no value given for y when $x = 1$, and that there is a hole or break in the graph at $x = 1$.

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Estimate the limit: $\lim_{x \to 2} \frac{x^3 - 2x^2 + 3x - 6}{x - 2}$.

EXAMPLE 5 Using a Graph to Find a Limit

Find the limit of $f(x)$ as x approaches 3.

$$f(x) = \begin{cases} 2, & x \neq 3 \\ 0, & x = 3 \end{cases}$$

Solution Because $f(x) = 2$ for all x other than $x = 3$ and the value of $f(3)$ is immaterial, it follows that the limit is 2 (see Figure 12.3). So, write

$$\lim_{x \to 3} f(x) = 2.$$

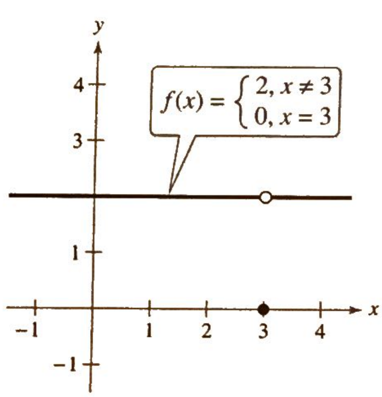

Figure 12.3

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Find the limit of $f(x)$ as x approaches 2.

$$f(x) = \begin{cases} -3, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

In Example 5, the fact that $f(3) = 0$ has no bearing on the existence or value of the limit as x approaches 3. For example, if the function were defined as

$$f(x) = \begin{cases} 2, & x \neq 3 \\ 4, & x = 3 \end{cases}$$

then the limit as x approaches 3 would still equal 2.

Limits That Fail to Exist

Next, you will examine some limits that fail to exist.

EXAMPLE 6 **Comparing Left and Right Behavior**

Show that the limit does not exist.

$$\lim_{x\to 0} \frac{|x|}{x}$$

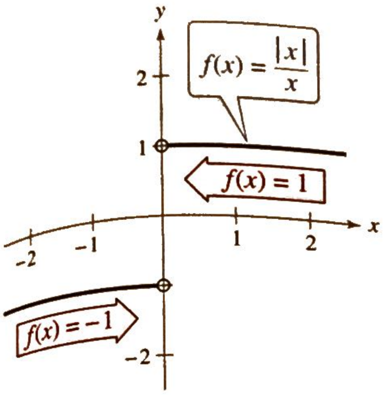

Figure 12.4

Solution Consider the graph of $f(x) = |x|/x$, shown in Figure 12.4. Notice that for positive x-values

$$\frac{|x|}{x} = 1, \quad x > 0$$

and for negative x-values

$$\frac{|x|}{x} = -1, \quad x < 0.$$

This means that no matter how close x gets to 0, there are both positive and negative x-values that yield $f(x) = 1$ and $f(x) = -1$, respectively. This implies that the limit does not exist.

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Show that the limit does not exist.

$$\lim_{x\to 1} \frac{2|x-1|}{x-1}$$

EXAMPLE 7 **Unbounded Behavior**

Discuss the existence of the limit.

$$\lim_{x\to 0} \frac{1}{x^2}$$

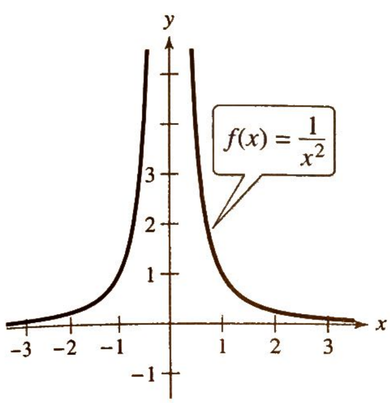

Figure 12.5

Solution Let $f(x) = 1/x^2$. In Figure 12.5, note that as x approaches 0 from either the right or the left, $f(x)$ increases without bound. This means that choosing x close enough to 0 enables you to force $f(x)$ to be as large as you want. For example, $f(x)$ is larger than 100 when you choose x that is within $\frac{1}{10}$ of 0. That is,

$$0 < |x| < \frac{1}{10} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 100.$$

Similarly, you can force $f(x)$ to be larger than 1,000,000 by choosing x that is within $\frac{1}{1000}$ of 0, as shown below.

$$0 < |x| < \frac{1}{1000} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 1{,}000{,}000$$

Because $f(x)$ is not approaching a unique real number L as x approaches 0, the limit does not exist.

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Discuss the existence of the limit.

$$\lim_{x\to 0} \left(-\frac{1}{x^2}\right)$$

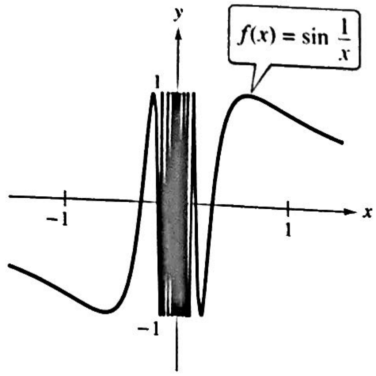

Figure 12.6

EXAMPLE 8 Oscillating Behavior

See LarsonPrecalculus.com for an interactive version of this type of example.

Discuss the existence of the limit.

$$\lim_{x\to 0} \sin \frac{1}{x}$$

Solution Let $f(x) = \sin(1/x)$. Notice in Figure 12.6 that as x approaches 0, $f(x)$ oscillates between -1 and 1. So, the limit does not exist because no matter how close you are to 0, it is possible to choose values of x_1 and x_2 such that $\sin(1/x_1) = 1$ and $\sin(1/x_2) = -1$, as shown in the table.

x	$-\frac{2}{\pi}$	$-\frac{2}{3\pi}$	$-\frac{2}{5\pi}$	0	$\frac{2}{5\pi}$	$\frac{2}{3\pi}$	$\frac{2}{\pi}$
$\sin \frac{1}{x}$	-1	1	-1	?	1	-1	1

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Discuss the existence of the limit.

$$\lim_{x\to 1} \cos \frac{1}{x-1}$$

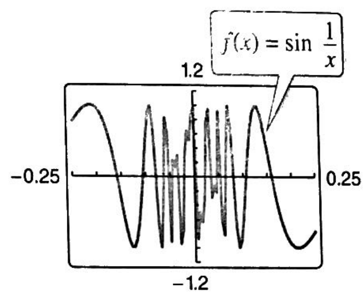
Examples 6, 7, and 8 show three of the most common types of behavior associated with the *nonexistence* of a limit.

Conditions Under Which Limits Do Not Exist

The limit of $f(x)$ as $x \to c$ does not exist when any of the conditions listed below are true.

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side of c. — Example 6
2. $f(x)$ increases or decreases without bound as x approaches c. — Example 7
3. $f(x)$ oscillates between two fixed values as x approaches c. — Example 8

TECHNOLOGY A graphing utility can help you discover the behavior of a function near the x-value at which you are evaluating a limit. When you do this, however, realize that you should not always trust the graphs that graphing utilities display. For instance, when you use a graphing utility to graph the function in Example 8 over an interval containing 0, you will most likely obtain an incorrect graph, as shown at the right. The reason that a graphing utility cannot show the correct graph is that the graph has infinitely many oscillations over any interval that contains 0.



Properties of Limits and Direct Substitution

Sometimes, as in Example 2, the limit of $f(x)$ as $x \to c$ is $f(c)$. In such cases, the limit can be evaluated by **direct substitution.** That is,

$$\lim_{x \to c} f(x) = f(c). \qquad \text{Substitute } c \text{ for } x.$$

There are many "well-behaved" functions, such as polynomial functions and rational functions with nonzero denominators, that have this property. The list below includes some basic limits.

> **Basic Limits**
>
> Let b and c be real numbers and let n be a positive integer.
>
> 1. $\lim_{x \to c} b = b$ — Limit of a constant function
> 2. $\lim_{x \to c} x = c$ — Limit of the identity function
> 3. $\lim_{x \to c} x^n = c^n$ — Limit of a power function
> 4. $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$, valid for all c when n is odd and valid for $c > 0$ when n is even — Limit of a radical function

For a proof of the limit of a power function, see Proofs in Mathematics on page 874.

This list can also include trigonometric functions. For example,

$$\lim_{x \to \pi} \sin x = \sin \pi = 0$$

and

$$\lim_{x \to 0} \cos x = \cos 0 = 1.$$

By combining the basic limits listed above with the properties of limits listed below, you can find limits for a wide variety of functions.

> **Properties of Limits**
>
> Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the limits
>
> $$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K.$$
>
> 1. Scalar multiple: $\lim_{x \to c} [bf(x)] = bL$
> 2. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$
> 3. Product: $\lim_{x \to c} [f(x)g(x)] = LK$
> 4. Quotient: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}, \quad K \neq 0$
> 5. Power: $\lim_{x \to c} [f(x)]^n = L^n$

TECHNOLOGY When evaluating limits, remember that there are several ways to solve most problems. Often, a problem can be solved *numerically, graphically,* or *algebraically.* A graphing utility can be used to confirm limits numerically with the *table* feature or graphically with the *zoom* and *trace* features.

EXAMPLE 9 Direct Substitution and Properties of Limits

Find each limit.

a. $\lim_{x \to 4} x^2$ **b.** $\lim_{x \to 4} 5x$ **c.** $\lim_{x \to \pi} \frac{\tan x}{x}$

d. $\lim_{x \to 9} \sqrt{x}$ **e.** $\lim_{x \to \pi} (x \cos x)$ **f.** $\lim_{x \to 3} (x + 4)^2$

Solution Use the properties of limits and direct substitution to evaluate each limit.

a. $\lim_{x \to 4} x^2 = (4)^2 = 16$ Use direct substitution.

b. $\lim_{x \to 4} 5x = 5 \lim_{x \to 4} x = 5(4) = 20$ Use the Scalar Multiple Property and direct substitution.

c. $\lim_{x \to \pi} \frac{\tan x}{x} = \frac{\lim_{x \to \pi} \tan x}{\lim_{x \to \pi} x} = \frac{\tan \pi}{\pi} = \frac{0}{\pi} = 0$ Use the Quotient Property and direct substitution.

d. $\lim_{x \to 9} \sqrt{x} = \sqrt{9} = 3$ Use direct substitution.

e. $\lim_{x \to \pi} (x \cos x) = \left(\lim_{x \to \pi} x\right)\left(\lim_{x \to \pi} \cos x\right) = \pi(\cos \pi) = -\pi$ Use the Product Property and direct substitution.

f. $\lim_{x \to 3} (x + 4)^2 = \left[\left(\lim_{x \to 3} x\right) + \left(\lim_{x \to 3} 4\right)\right]^2 = (3 + 4)^2 = 49$ Use the Power and Sum Properties and direct substitution.

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Find each limit.

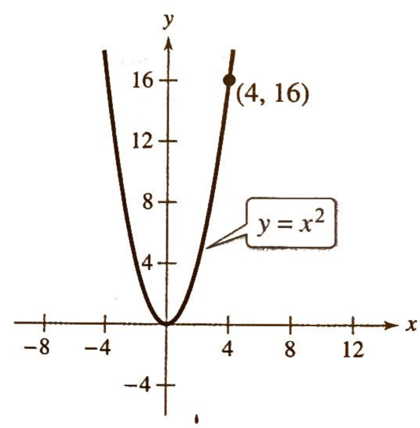
a. $\lim_{x \to 3} \frac{1}{4}$ **b.** $\lim_{x \to 3} x^3$ **c.** $\lim_{x \to \pi} \frac{\cos x}{x}$

d. $\lim_{x \to 12} \sqrt{x}$ **e.** $\lim_{x \to \pi} (x \tan x)$ **f.** $\lim_{x \to 3} (1 - x)^2$

Example 9 shows algebraic solutions. To verify the limit in Example 9(a) numerically, create a table that shows values of x^2 for two sets of x-values—one set that approaches 4 from the left and one that approaches 4 from the right.

x	3.9	3.99	3.999	4.0	4.001	4.01	4.1
x^2	15.2100	15.9201	15.9920	?	16.0080	16.0801	16.8100

The table shows that the limit as x approaches 4 is 16. To verify the limit in Example 9(a) graphically, sketch the graph of $y = x^2$, as shown below. The graph also shows that the limit as x approaches 4 is 16.



The results of using direct substitution to evaluate limits of polynomial and rational functions are summarized below.

Limits of Polynomial and Rational Functions

1. If p is a polynomial function and c is a real number, then
$$\lim_{x \to c} p(x) = p(c).$$
2. If r is a rational function $r(x) = p(x)/q(x)$, and c is a real number such that $q(c) \neq 0$, then
$$\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

For a proof of the limit of a polynomial function, see Proofs in Mathematics on page 874.

EXAMPLE 10 **Evaluating Limits by Direct Substitution**

Find each limit.

a. $\lim_{x \to -1} (x^2 + x - 6)$ **b.** $\lim_{x \to -1} \frac{x^3 - 5x}{x}$ **c.** $\lim_{x \to -1} \frac{x^2 + x - 6}{x + 3}$

Solution The first function is a polynomial function. The second and third functions are rational functions (with nonzero denominators at $x = -1$). So, you can evaluate the limits by direct substitution.

a. $\lim_{x \to -1} (x^2 + x - 6) = (-1)^2 + (-1) - 6 = -6$

b. $\lim_{x \to -1} \frac{x^3 - 5x}{x} = \frac{(-1)^3 - 5(-1)}{-1} = -\frac{4}{1} = -4$

c. $\lim_{x \to -1} \frac{x^2 + x - 6}{x + 3} = \frac{(-1)^2 + (-1) - 6}{-1 + 3} = -\frac{6}{2} = -3$

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Find each limit.

a. $\lim_{x \to 3} (x^2 - 3x + 7)$ **b.** $\lim_{x \to 3} \frac{x^2 - 3x + 7}{x}$ **c.** $\lim_{x \to 3} \frac{x + 3}{x^2 + 3x}$

Summarize **(Section 12.1)**

1. State two uses of the limit concept in calculus *(page 818)*. For an example that uses limit terminology to state a maximum area, see Example 1.
2. State the definition of a limit *(page 819)*. For examples of estimating limits and using graphs to find limits, see Examples 2–5.
3. List the three most common types of behavior associated with the nonexistence of a limit *(pages 821 and 822)*. For examples of functions that do not have limits, see Examples 6–8.
4. Explain how to use properties of limits and direct substitution to evaluate a limit *(page 823)*. For examples of using properties of limits and direct substitution to evaluate limits, see Examples 9 and 10.