12.2 Techniques for Evaluating Limits

Limits have applications in real-life situations. For instance, in Exercises 41 and 42 on page 838, you will use limits to find the velocity of a free-falling object at different times.

- **Use the dividing out technique to evaluate limits of functions.**
- **Use the rationalizing technique to evaluate limits of functions.**
- **Use technology to approximate limits of functions numerically and graphically.**
- **Evaluate one-sided limits of functions.**
- **Evaluate limits from calculus.**

Dividing Out Technique

In Section 12.1, you studied several types of functions whose limits can be evaluated by direct substitution. In this section, you will study several techniques for evaluating limits of functions for which direct substitution fails. For example, consider the limit

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}.$$

Direct substitution produces 0 in both the numerator and denominator.

$$(-3)^2 + (-3) - 6 = 0 \qquad \text{Numerator is 0 when } x = -3.$$

$$-3 + 3 = 0 \qquad \text{Denominator is 0 when } x = -3.$$

The resulting fraction, $\frac{0}{0}$, has no meaning as a real number. It is called an **indeterminate form** because you cannot, from the form alone, determine the limit. By using a table, however, it appears that the limit of the function as x approaches -3 is -5.

x	-3.01	-3.001	-3.0001	-3	-2.9999	-2.999	-2.99
$\frac{x^2 + x - 6}{x + 3}$	-5.01	-5.001	-5.0001	?	-4.9999	-4.999	-4.99

When you attempt to evaluate a limit of a rational function by direct substitution and encounter the indeterminate form $\frac{0}{0}$, the numerator and denominator must have a common factor. After factoring and dividing out, use direct substitution again. Examples 1 and 2 show this **dividing out technique.**

EXAMPLE 1 Dividing Out Technique

See LarsonPrecalculus.com for an interactive version of this type of example.

Find the limit: $\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$.

Solution From the discussion above, you know that direct substitution fails. So, begin by factoring the numerator and dividing out any common factors.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3} = \lim_{x \to -3} \frac{(x - 2)(x + 3)}{x + 3} \qquad \text{Factor numerator.}$$

$$= \lim_{x \to -3} (x - 2) \qquad \text{Divide out common factor and simplify.}$$

$$= -5 \qquad \text{Direct substitution}$$

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Find the limit: $\lim_{x \to 4} \frac{x^2 - 7x + 12}{x - 4}$.

SoleilC /Shutterstock.com

The validity of the dividing out technique stems from the fact that when two functions agree at all but a single number c, they must have identical limit behavior at $x = c$. In Example 1, the functions

$$f(x) = \frac{x^2 + x - 6}{x + 3} \quad \text{and} \quad g(x) = x - 2$$

agree at all values of x other than $x = -3$. So, you can use $g(x)$ to find the limit of $f(x)$.

EXAMPLE 2 Dividing Out Technique

Find the limit.

$$\lim_{x \to 1} \frac{x - 1}{x^3 - x^2 + x - 1}$$

Solution Begin by substituting $x = 1$ into the numerator and denominator.

$1 - 1 = 0$ Numerator is 0 when $x = 1$.

$1^3 - 1^2 + 1 - 1 = 0$ Denominator is 0 when $x = 1$.

Both the numerator and denominator are zero when $x = 1$, so direct substitution will not yield the limit. To find the limit, factor the numerator and denominator, divide out any common factors, and then use direct substitution again.

$$\lim_{x \to 1} \frac{x - 1}{x^3 - x^2 + x - 1} = \lim_{x \to 1} \frac{x - 1}{(x - 1)(x^2 + 1)} \quad \text{Factor denominator.}$$

$$= \lim_{x \to 1} \frac{\cancel{x - 1}}{\cancel{(x - 1)}(x^2 + 1)} \quad \text{Divide out common factor.}$$

$$= \lim_{x \to 1} \frac{1}{x^2 + 1} \quad \text{Simplify.}$$

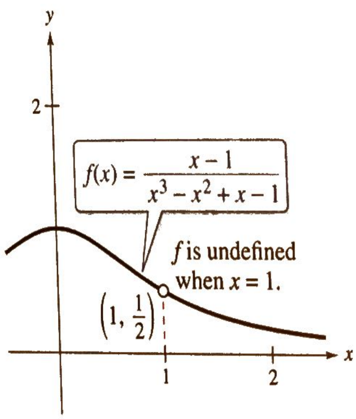
$$= \frac{1}{1^2 + 1} \quad \text{Direct substitution}$$

$$= \frac{1}{2} \quad \text{Simplify.}$$

REMARK To obtain the factorization of the denominator, divide by $(x - 1)$ or factor by grouping.

$$x^3 - x^2 + x - 1 = x^2(x - 1) + (x - 1) = (x - 1)(x^2 + 1)$$

The graph below verifies this result.



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Find the limit: $\lim_{x \to 7} \frac{x - 7}{x^3 - 7x^2 + 7x - 49}$.

ALGEBRA HELP To review techniques for rationalizing numerators and denominators, see Appendix A.2.

Rationalizing Technique

A way to find the limits of some functions is to first rationalize the numerator of the function. This is the **rationalizing technique.** Recall that to rationalize a numerator of the form $a \pm b\sqrt{m}$ or $b\sqrt{m} \pm a$, multiply the numerator and denominator by the *conjugate* of the numerator. For example, the conjugate of $\sqrt{x} + 4$ is $\sqrt{x} - 4$.

EXAMPLE 3 **Rationalizing Technique**

Find the limit.

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$$

Solution By direct substitution, you obtain the indeterminate form $\frac{0}{0}$.

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \frac{\sqrt{0+1}-1}{0} = \frac{0}{0} \qquad \text{Indeterminate form}$$

In this case, rewrite the fraction by rationalizing the numerator.

$$\frac{\sqrt{x+1}-1}{x} = \left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)$$

$$= \frac{(x+1)-1}{x\left(\sqrt{x+1}+1\right)} \qquad \text{Multiply.}$$

$$= \frac{x}{x\left(\sqrt{x+1}+1\right)} \qquad \text{Simplify.}$$

$$= \frac{\cancel{x}}{\cancel{x}\left(\sqrt{x+1}+1\right)} \qquad \text{Divide out common factor.}$$

$$= \frac{1}{\sqrt{x+1}+1}, \quad x \neq 0 \qquad \text{Simplify.}$$

Now, evaluate the limit by direct substitution.

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} = \frac{1}{\sqrt{0+1}+1} = \frac{1}{1+1} = \frac{1}{2}$$

To verify your conclusion that the limit is $\frac{1}{2}$, construct a table, such as the one shown below, or sketch a graph, as shown in Figure 12.7.

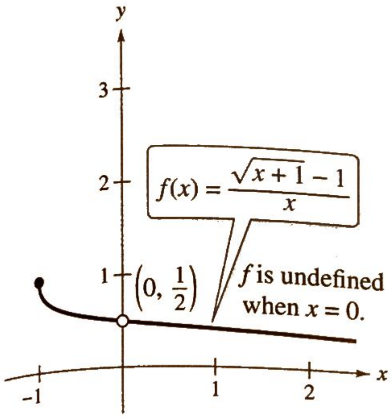

Figure 12.7

x	−0.1	−0.01	−0.001	0	0.001	0.01	0.1
$f(x)$	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881

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Find the limit.

$$\lim_{x\to 0} \frac{1-\sqrt{1-x}}{x}$$

The rationalizing technique for evaluating limits is based on multiplication by a convenient form of 1. In Example 3, the convenient form is

$$1 = \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}.$$

Using Technology

The dividing out and rationalizing techniques may not work when finding limits of nonalgebraic functions. You often need to use more sophisticated analytic techniques to find limits of nonalgebraic functions, as shown in Examples 4 and 5.

EXAMPLE 4 Approximating a Limit

Approximate the limit.

$$\lim_{x\to 0} (1 + x)^{1/x}$$

Numerical Solution

Let $f(x) = (1 + x)^{1/x}$. Use the *table* feature of a graphing utility to create a table that shows the values of f for x starting at $x = -0.003$ with a step of 0.001, as shown in the figure below. Because 0 is halfway between -0.001 and 0.001, use the average of the values of f at these two x-values to estimate the limit.

$$\lim_{x\to 0} (1 + x)^{1/x} \approx \frac{2.7196 + 2.7169}{2} = 2.71825$$

The actual limit can be found algebraically to be $e \approx 2.71828$.

X	Y1
-.003	2.7224
-.002	2.721
-.001	2.7196
0	ERROR
.001	2.7169
.002	2.7156
.003	2.7142
X=0	

Graphical Solution

To approximate the limit graphically, graph the function $f(x) = (1 + x)^{1/x}$, as shown in Figure 12.8. Using the *zoom* and *trace* features of the graphing utility, choose two points on the graph of f, such as

$$(-0.00017, 2.7185) \quad \text{and} \quad (0.00017, 2.7181)$$

as shown in Figure 12.9. The x-coordinates of these two points are equidistant from 0, so approximate the limit to be the average of the y-coordinates. That is,

$$\lim_{x\to 0} (1 + x)^{1/x} \approx \frac{2.7185 + 2.7181}{2} = 2.7183.$$

The actual limit can be found algebraically to be $e \approx 2.71828$.

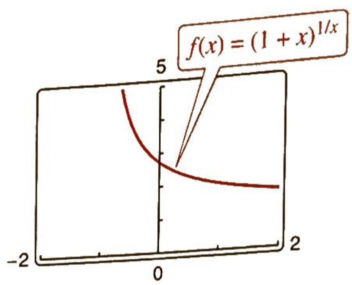

Figure 12.8

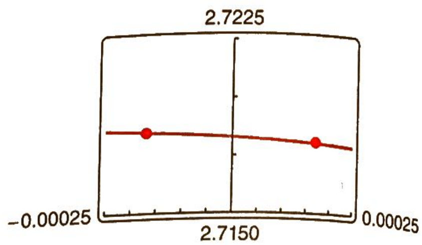

Figure 12.9

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Approximate the limit: $\lim_{x\to 0} \dfrac{e^x - 1}{x}$.

EXAMPLE 5 Approximating a Limit Graphically

Approximate the limit: $\lim_{x\to 0} \dfrac{\sin x}{x}$.

Solution Direct substitution produces the indeterminate form $\frac{0}{0}$. To approximate the limit, begin by using a graphing utility to graph $f(x) = (\sin x)/x$, as shown in Figure 12.10. Then use the *zoom* and *trace* features of the graphing utility to choose a point on each side of 0, such as $(-0.0012467, 0.9999997)$ and $(0.0012467, 0.9999997)$. Finally, approximate the limit as the average of the y-coordinates of these two points, $\lim_{x\to 0} (\sin x)/x \approx 0.9999997$. Example 10 shows that this limit is exactly 1.

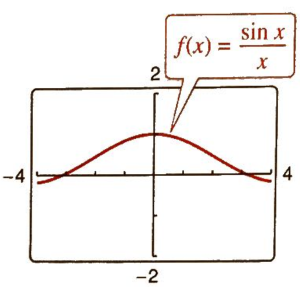

Figure 12.10

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Approximate the limit: $\lim_{x\to 0} \dfrac{1 - \cos x}{x}$.

TECHNOLOGY The graphs shown in Figures 12.8 and 12.10 appear to be continuous at $x = 0$. However, when you use the *trace* or the *value* feature of a graphing utility to determine $f(0)$, no value is given. Some graphing utilities show breaks or holes in a graph when you use an appropriate viewing window. The holes in the graphs in Figures 12.8 and 12.10 occur on the y-axis, so the holes are not visible.

One-Sided Limits

In Section 12.1, you saw that one way in which a limit can fail to exist is when a function approaches a different value from the left side of c than it approaches from the right side of c. This type of behavior can be described more concisely with the concept of a **one-sided limit.**

$$\lim_{x\to c^-} f(x) = L_1 \text{ or } f(x)\to L_1 \text{ as } x\to c^- \qquad \text{Limit from the left}$$

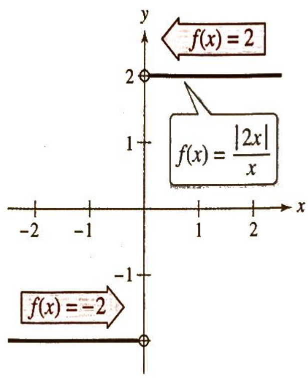
$$\lim_{x\to c^+} f(x) = L_2 \text{ or } f(x)\to L_2 \text{ as } x\to c^+ \qquad \text{Limit from the right}$$

EXAMPLE 6 Evaluating One-Sided Limits

Find the limit of $f(x)$ as x approaches 0 from the left and from the right.

$$f(x) = \frac{|2x|}{x}$$

Solution From the graph of f, shown below, notice that $f(x) = -2$ for all $x < 0$.



So, the limit from the left is

$$\lim_{x\to 0^-} \frac{|2x|}{x} = -2. \qquad \text{Limit from the left: } f(x)\to -2 \text{ as } x\to 0^-$$

Also from the graph, notice that $f(x) = 2$ for all $x > 0$, so the limit from the right is

$$\lim_{x\to 0^+} \frac{|2x|}{x} = 2. \qquad \text{Limit from the right: } f(x)\to 2 \text{ as } x\to 0^+$$

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Find the limit of $f(x)$ as x approaches 3 from the left and from the right.

$$f(x) = \frac{|x-3|}{x-3}$$

In Example 6, note that the function approaches different limits from the left and from the right. In such cases, the limit of $f(x)$ as $x\to c$ does not exist. For the limit of a function to exist as $x\to c$, it must be true that both one-sided limits exist and are equal.

Existence of a Limit

If f is a function and c and L are real numbers, then

$$\lim_{x\to c} f(x) = L$$

if and only if both the left and right limits *exist* and are *equal* to L.

EXAMPLE 7 Evaluating One-Sided Limits

Find the limit of $f(x)$ as x approaches 1.

$$f(x) = \begin{cases} 4 - x, & x < 1 \\ 4x - x^2, & x > 1 \end{cases}$$

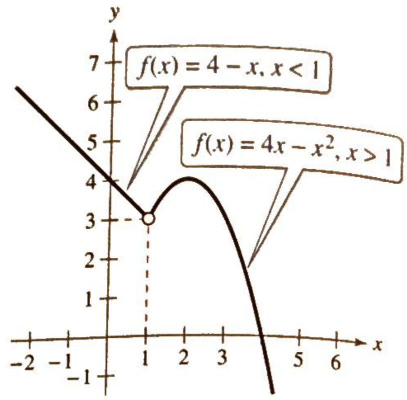
Solution Remember that you are concerned about the value of f *near* $x = 1$ rather than *at* $x = 1$. So, for $x < 1$, $f(x)$ is given by $4 - x$. Use direct substitution to obtain

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (4 - x) = 4 - 1 = 3.$$

For $x > 1$, $f(x)$ is given by $4x - x^2$. Use direct substitution to obtain

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x - x^2) = 4(1) - 1^2 = 3.$$

Both one-sided limits exist and are equal to 3, so it follows that $\lim_{x \to 1} f(x) = 3$. The graph at the right confirms this conclusion.



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Find the limit of $f(x)$ as x approaches -1.

$$f(x) = \begin{cases} -x^2 - 3x, & x < -1 \\ x + 3, & x > -1 \end{cases}$$

EXAMPLE 8 Comparing Limits from the Left and Right

For 2-day shipping, a delivery service charges \$24 for the first pound and \$4 for each additional pound or portion of a pound. Let x represent the weight (in pounds) of a package and let $f(x)$ represent the shipping cost. Show that the limit of $f(x)$ as $x \to 2$ does not exist.

$$f(x) = \begin{cases} \$24, & 0 < x \le 1 \\ \$28, & 1 < x \le 2 \\ \$32, & 2 < x \le 3 \end{cases}$$

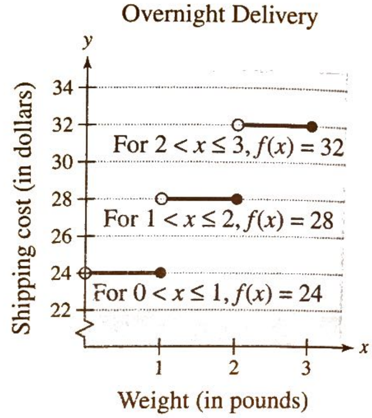
Solution The graph of f is at the right. The limit of $f(x)$ as x approaches 2 from the left is

$$\lim_{x \to 2^-} f(x) = 28$$

whereas the limit of $f(x)$ as x approaches 2 from the right is

$$\lim_{x \to 2^+} f(x) = 32.$$

These one-sided limits are not equal, so the limit of $f(x)$ as $x \to 2$ does not exist.



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In Example 8, show that the limit of $f(x)$ as $x \to 1$ does not exist.

As illustrated in Example 8, a *step function* can model delivery charges. The limit of such a function does not exist at a "jump" because the one-sided limits are not equal.

ALGEBRA HELP To review evaluating difference quotients, see Section 1.4.

Limits from Calculus

In the next example, you will study an important type of limit from calculus—the limit of a *difference quotient.*

EXAMPLE 9 **Evaluating a Limit from Calculus**

For the function

$$f(x) = x^2 - 1$$

find

$$\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}.$$

Solution Begin by substituting for $f(3 + h)$ and $f(3)$ and simplifying.

$$\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \to 0} \frac{[(3 + h)^2 - 1] - (3^2 - 1)}{h}$$

$$= \lim_{h \to 0} \frac{9 + 6h + h^2 - 1 - 9 + 1}{h}$$

$$= \lim_{h \to 0} \frac{6h + h^2}{h}$$

By factoring and dividing out, you obtain

$$\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \to 0} \frac{\cancel{h}(6 + h)}{\cancel{h}}$$

$$= \lim_{h \to 0} (6 + h)$$

$$= 6 + 0$$

$$= 6.$$

So, the limit is 6.

REMARK Note that for any x-value, the limit of a difference quotient is an expression of the form

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

Direct substitution into the difference quotient always produces the indeterminate form $\frac{0}{0}$.

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \frac{f(x + 0) - f(x)}{0}$$

$$= \frac{f(x) - f(x)}{0}$$

$$= \frac{0}{0}$$

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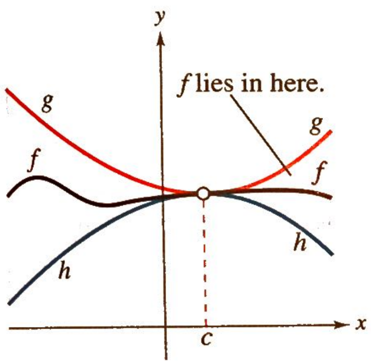
For the function $f(x) = 4x - x^2$, find

$$\lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}.$$

The theorem below concerns the limit of a function that is squeezed between two other functions, each of which has the same limit at a given x-value, as shown in Figure 12.11.

$h(x) \le f(x) \le g(x)$



The Squeeze Theorem
Figure 12.11

> **The Squeeze Theorem**
>
> If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if
>
> $$\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$$
>
> then $\lim_{x \to c} f(x)$ exists and is equal to L.

A proof of the Squeeze Theorem is beyond the scope of this text, but you may be able to see, intuitively, that this theorem makes sense. Example 10 shows an application of the Squeeze Theorem.

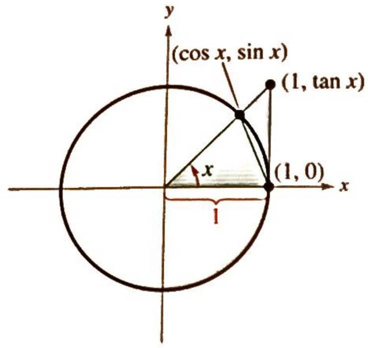


Figure 12.12

REMARK Recall from Section 4.1 that the area of a sector of a circle of radius r is given by

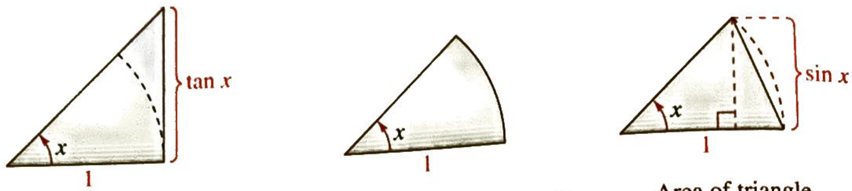
$$A = \frac{1}{2}r^2\theta$$

where θ is the measure of the central angle in radians.

EXAMPLE 10 A Special Trigonometric Limit

Find the limit: $\lim_{x \to 0} \dfrac{\sin x}{x}$.

Solution One way to find this limit is to consider a sector of a circle of radius 1 with central angle x, "squeezed" between two triangles (see Figure 12.12), where x is an acute positive angle measured in radians.



Area of triangle		Area of sector		Area of triangle
$\dfrac{\tan x}{2}$	$\geq$	$\dfrac{x}{2}$	$\geq$	$\dfrac{\sin x}{2}$

Multiplying each expression by $2/\sin x$ produces

$$\frac{1}{\cos x} \geq \frac{x}{\sin x} \geq 1$$

and taking reciprocals and reversing the inequality symbols yields

$$\cos x \leq \frac{\sin x}{x} \leq 1.$$

Because $\cos x = \cos(-x)$ and $(\sin x)/x = [\sin(-x)]/(-x)$, this inequality is valid for *all* nonzero x in the open interval $(-\pi/2, \pi/2)$. Finally, $\lim_{x \to 0} \cos x = 1$ and $\lim_{x \to 0} 1 = 1$, so apply the Squeeze Theorem to conclude that $\lim_{x \to 0} (\sin x)/x = 1$.

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Find the limit: $\lim_{x \to 0} \dfrac{1 - \cos x}{x}$.

Summarize (Section 12.2)

1. Explain how to use the dividing out technique to evaluate the limit of a function *(page 829)*. For examples of the dividing out technique, see Examples 1 and 2.
2. Explain how to use the rationalizing technique to evaluate the limit of a function *(page 831)*. For an example of the rationalizing technique, see Example 3.
3. Explain how to use technology to approximate limits of functions numerically and graphically *(page 832)*. For examples of using technology to approximate limits, see Examples 4 and 5.
4. Explain the concept of a one-sided limit and how to use one-sided limits to determine the existence of a limit *(page 833)*. For examples of evaluating one-sided limits, see Examples 6–8.
5. Explain how to evaluate the limit of a difference quotient and how to evaluate a limit using the Squeeze Theorem *(page 835)*. For examples of evaluating such limits from calculus, see Examples 9 and 10.